

The University of Chicago

MOTION OF A SPHEROID AND A
SPHERE UNDER MUTUAL
GRAVITATION

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

AUGUST, 1940

By
SAMUEL SCHWARTZ SMITH

CHICAGO, ILLINOIS

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MOTION OF A SPHEROID AND A SPHERE
UNDER MUTUAL GRAVITATION

1. Introduction.—It seems that nearly every discussion that pertains to the behavior of two material bodies under their mutual gravitation traces its origin back to Sir Isaac Newton, who in his Principia gives a complete treatment of the motion of two homogeneous spheres. For many astronomical applications it is necessary to take into account departures from sphericity of the attracting bodies. Thus, by taking into account oblateness, Tisserand in his Traité De Mécanique Céleste determined the rate of change of the earth's angular momentum due to the attraction of the sun and moon. Here we shall first consider the general motion of any two rigid bodies which are free to move in space and are only acted upon by their mutual attraction.

The equations of motion are obtained by employing the notions of vectors and dyadics as developed by Gibbs,¹ who was the first to introduce vectors into astronomy. He applied them to the solution of the two-body problem and also in the determination of orbits. This notation greatly simplifies the derivation of the equations and is in contrast with that employed by Tisserand, who used the generalized co-ordinates of Lagrange. After the equations of motions are derived for the two bodies further investigations are limited to the motion of a spheroid and a sphere. In this case, by means of algebraic integrals, the equations of relative motions are expressed in terms of the elements as the only variables.

The case of a spheroid rotating on its principal axis and a particle was treated by MacMillan.² However, with no restriction of the spheroid, it is possible to find an integral which resembles Jacobi's integral in the three-body problem, and which gives the

¹E. B. Wilson, Vector Analysis, Gibbs (New Haven: Yale University Press, 1925).

²F. R. Moulton, Periodic Orbits (Washington: Carnegie Institution, 1920), Publication No. 161, chap. iv, p. 99.

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value of the velocity squared in terms of co-ordinates relative to the body. This integral permits us to define surfaces of zero relative velocity.

2. The Potential Function of Two Rigid Bodies.—Consider a system of two distinct rigid bodies which are free to move in space and are only acted upon by their mutual attractions. Let the two bodies be denoted by M_1 and M_2 , and x_1, y_1, z_1 be the co-ordinates of an element of mass dM_1 of body M_1 relative to any co-ordinate axes. Then

$$dM_1 = \sigma_1 dx_1 dy_1 dz_1,$$

where σ_1 is the density of the body M_1 at the point x_1, y_1, z_1 .

Let a like set of symbols but with subscript 2 be used for the body M_2 . Then the potential of the mass of M_2 with reference to the mass of M_1 is the scalar function of position³

$$(2:1) \quad V = k^2 \int_{M_1} \int_{M_2} \frac{dM_1 dM_2}{r_{12}},$$

$$r_{12}^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2,$$

the integration being extended over the entire body M_1 and M_2 respectively, and k is the gravitational constant. If the density is continuous, the function exists since the integrand is continuous over the region of integration.

To evaluate the potential function V let O_1 be the center of mass of the body M_1 and the origin of the co-ordinate system in M_1 ; O_2 be the center of mass of the body M_2 and the origin of the co-ordinate system in M_2 ; R be a vector with the origin at O_1 and terminating at O_2 ; R_1 be a vector from O_1 to the element dM_1 of the body M_1 ; and R_2 be a vector from O_2 to the element dM_2 of the body M_2 .

³B. O. Pierce, Elements of the Theory of Newtonian Potential Function (New York: Ginn & Co., 1902).
W. D. MacMillan, The Theory of the Potential (New York: McGraw-Hill Co., 1930).

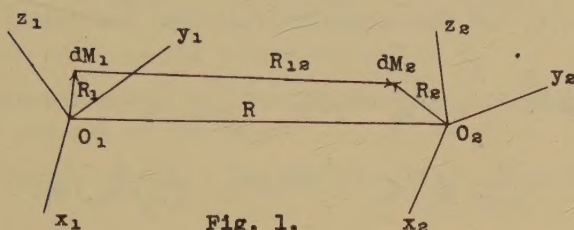


Fig. 1.

If R_{12} is a vector from dM_1 to dM_2 then

$$(2.2) \quad R_{12} = R + R_2 - R_1.$$

The magnitude of the vector R_{12} is

$$r_{12} = \sqrt{R_{12} \cdot R_{12}} = \sqrt{R^2 + 2R \cdot (R_2 - R_1) + (R_2 - R_1)^2}.$$

Expanding $1/r_{12}$ in a power series in $1/r$,

$$(2.3) \quad \begin{aligned} 1/r_{12} = [1 + (2R \cdot R_1 - R_1^2 - 2R \cdot R_2 + 2R_1 \cdot R_2 - R_2^2)/2r^2 \\ + 3(2R \cdot R_1 - R_1^2 - 2R \cdot R_2 + 2R_1 \cdot R_2 - R_2^2)^2/8r^4 \\ + \dots]/r. \end{aligned}$$

The expression (2.3) is absolutely and uniformly convergent⁴ for

$$|R_2 - R_1| \leq H < R$$

and can be integrated term by term as long as the two bodies are far enough apart so that R meets this requirement, which is true as long as the bodies do not contact each other. This expansion may be expressed in the form

$$(2.4) \quad \begin{aligned} 1/r_{12} = [1 + (r_1/r) \cos \alpha_1 - r_1^2/2r^2 + (3r_1^2/2r^2) \cos^2 \alpha_1 \\ - (r_2/r) \cos \alpha_2 - r_2^2/2r^2 + (3r_2^2/2r^2) \cos^2 \alpha_2 \\ - (3r_1 r_2/2r^2) \cos \alpha_1 \cos \alpha_2 + (1/r^2) R_1 \cdot R_2 \\ + \dots]/r, \end{aligned}$$

where α_1 is the angle between R and R_1 and α_2 is the angle between R and R_2 . It includes all third order terms, neglecting those which are divisible by r^4 or higher powers of r .

⁴Ibid., p. 83.

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When the substitution for $1/r_{12}$ of (2.4) is made in the integral (2.1) for the potential function, the latter becomes

$$\begin{aligned}
 (2.5) \quad V = \frac{k^2}{r} & \left[\int_{M_1} \int_{M_2} dM_1 dM_2 + \frac{1}{r} \int_{M_1} \int_{M_2} r_1 \cos \alpha_1 dM_1 dM_2 \right. \\
 & - \frac{1}{2r^2} \int_{M_1} \int_{M_2} r_1^2 dM_1 dM_2 + \frac{3}{2r^2} \int_{M_1} \int_{M_2} r_1^2 \cos \alpha_1 dM_1 dM_2 \\
 & - \frac{1}{r} \int_{M_1} \int_{M_2} r_2 \cos \alpha_2 dM_1 dM_2 - \frac{1}{2r^2} \int_{M_1} \int_{M_2} r_2^2 dM_1 dM_2 \\
 & + \frac{3}{2r^2} \int_{M_1} \int_{M_2} r_2^2 \cos^2 \alpha_2 dM_1 dM_2 \\
 & - \frac{3}{2r^2} \int_{M_1} \int_{M_2} r_1 r_2 \cos \alpha_1 \cos \alpha_2 dM_1 dM_2 \\
 & \left. + \frac{1}{r^2} \int_{M_1} \int_{M_2} R_1 \cdot R_2 dM_1 dM_2 + \dots \right].
 \end{aligned}$$

The integrals in the right member of (2.5) are interpretable as follows:

$$\int_{M_1} \int_{M_2} dM_1 dM_2 = M_1 M_2;$$

$$\int_{M_1} \int_{M_2} r_p \cos \alpha_p dM_1 dM_2 = M_1 M_2 P_p = 0, \quad p = 1, 2,$$

where P_p is the projection of the vector from O_p to the center of mass M_p on R and is therefore zero;

$$\begin{aligned}
 \int_{M_1} \int_{M_2} r_p^2 \cos^2 \alpha_p dM_1 dM_2 &= \int_{M_1} \int_{M_2} r_p^2 dM_1 dM_2 \\
 &\quad - \int_{M_1} \int_{M_2} r_p^2 \sin^2 \alpha_p dM_1 dM_2 \\
 &= M_p J_q - M_p I_q, \quad p, q = 1, 2 \\
 &\quad p \neq q,
 \end{aligned}$$

where J_q is the moment of inertia of M_q with respect to the origin O_q and I_q is the moment of inertia of M_q with respect to the line of centers r ; and

$$\int_{M_1} \int_{M_2} R_1 \cdot R_2 \, dM_1 dM_2 = \left(\int_{M_1} R_1 dM_1 \right) \cdot \left(\int_{M_2} R_2 dM_2 \right) = 0,$$

since the origin of R_p is the center of mass of M_p .

The expression for the potential function V given in equation (2.5) now reduces to the form

$$(2.6) \quad V = k^2 [M_1 M_2 + M_2 (2J_1 - 3I_1)/2r^2 + M_1 (2J_2 - 3I_2)/2r^2 + \dots] / r,$$

provided terms of the fourth and higher order in $1/r$ are omitted.

The inertia of any body M about a vector R may be conveniently expressed in terms of a dyadic as

$$(2.7) \quad I = (R \cdot \Phi \cdot R) / r^2.$$

The dyadic Φ in the Gibbs notation is in terms of the unit vectors i, j, k . That is

$$\Phi = Aii + Bjj + Ckk + D(jk + kj) + E(ki + ik) + F(ij + ji)$$

where A, B, C are equal to the moments of inertia about axes through O parallel to i, j, k respectively; and D, E, F are equal to the negative of the products of inertia, as usually defined, about axes through O parallel to j and k, k and i, i and j respectively.⁵ If the set of rectangular unit vectors i, j, k are chosen as the principal axes of inertia then the products of inertia vanish and

$$\Phi = Aii + Bjj + Ckk.$$

We shall denote the dyadic of inertia of the bodies M_1 and M_2 by Φ_1 and Φ_2 respectively. Similarly subscripts 1 and 2 will be used to differentiate between the two sets of principal axes and moments of inertia of the respective bodies.

On substituting the value of I from equation (2.7) in the equation (2.6), with the proper subscripts attached, the expression for the potential function takes the form

$$(2.8) \quad V = k^2 [M_1 M_2 + M_2 J_1 / r^2 - (3M_2 / 2r^4) R \cdot \Phi_1 \cdot R + M_1 J_2 / r^2 - (3M_1 / 2r^4) R \cdot \Phi_2 \cdot R + \dots] / r.$$

Equation (2.8) gives a convenient form of the potential function of the mass of M_2 with reference to the mass of M_1 . It

⁵C. E. Weatherburn, Advanced Vector Analysis (London: G. Bell & Sons, Ltd., 1928), p. 105.

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will be noticed that the potential function, as here given in terms of third order in $1/r$, depends on the masses of the bodies, the distance between their centers of mass, the moments of inertia about their centers of mass, and the moments of inertia about their line of centers with respect to the principal axes of the bodies. It is dependent on the shape of the body only in so far as the shape determines these moments of inertia. Consequently with this degree of approximation it is possible to replace the bodies with homogeneous ellipsoids.

3. Differential Equations for Motion of Center of Masses.—

The resultant of the attraction of the body M_1 on the body M_2 equals the gradient of V with respect to R . If this gradient is denoted by ∇V , then for any displacement dR of R the corresponding change in potential⁶ is

$$(3.1) \quad dV = dR \cdot \nabla V.$$

Now the differential of the potential function of equation (2.8)

$$(3.2) \quad dV = k^2 [-(M_1 M_2 / r^2) dr - (3M_2 J_1 / r^4) dr + (15M_2 / 2r^6) R \cdot \Phi_1 \cdot R dr \\ - (3M_2 / r^5) R \cdot \Phi_1 \cdot dR - (3M_1 J_2 / r^4) dr + (15M_1 / 2r^6) R \cdot \Phi_2 \cdot R dr \\ - (3M_1 / r^5) R \cdot \Phi_2 \cdot dR],$$

and since

$$r dr = R \cdot dR$$

equation (3.2) can be written as

$$(3.3) \quad dV = k^2 [-(M_1 M_2 / r^3) R - (3M_2 J_1 / r^5) R + (15M_2 / 2r^7) R \cdot \Phi_1 \cdot R R \\ - (3M_2 / r^5) R \cdot \Phi_1 - (3M_1 J_2 / r^5) R + (15M_1 / 2r^7) R \cdot \Phi_2 \cdot R R \\ - (3M_1 / r^5) R \cdot \Phi_2] \cdot dR.$$

Since (3.3) holds for all displacements dR it follows from (3.1) that the resultant of the attraction of M_1 on M_2 is

$$(3.4) \quad \nabla V = k^2 [-(M_1 M_2 / r^3) R - (3M_2 J_1 / r^5) R + (15M_2 / 2r^7) R \cdot \Phi_1 \cdot R R \\ - (3M_2 / r^5) R \cdot \Phi_1 - (3M_1 J_2 / r^5) R + (15M_1 / 2r^7) R \cdot \Phi_2 \cdot R R \\ - (3M_1 / r^5) R \cdot \Phi_2].$$

⁶Wilson, op. cit., p. 140.

If we take O as the center of mass of M_1 and M_2 and let in this section R_1 be the vector with origin at O and terminating at O_1 and R_2 be the vector with origin at O and terminating at O_2 , then

$$(3.5) \quad \begin{aligned} -R_1 + R_2 &= R, \\ R_1 M_1 + R_2 M_2 &= 0, \end{aligned}$$

and therefore,

$$(3.6) \quad \begin{aligned} R_2 &= [M_1/(M_1 + M_2)]R, \\ R''_2 &= [M_1/(M_1 + M_2)]R''. \end{aligned}$$

Now the equations of motion of M_1 and M_2 referred to their center of mass as origin can be expressed as

$$(3.7) \quad M_p R''_p = \nabla V, \quad p = 1, 2.$$

Hence by (3.6) the equation of relative motion is

$$(3.8) \quad \begin{aligned} MR'' &= k^2 [-(M_1 M_2 / r^3)R - (3M_2 J_1 / r^5)R + (15M_2 / 2r^7)R \cdot \hat{\Phi}_1 \cdot RR \\ &\quad - (M_2 / r^5)R \cdot \hat{\Phi}_1 - (3M_1 J_2 / r^5)R + (15M_1 / 2r^7)R \cdot \hat{\Phi}_2 \cdot RR \\ &\quad - (M_1 / r^5)R \cdot \hat{\Phi}_2], \end{aligned}$$

where M , with no subscript, shall be defined as the value of $M_1 M_2 / (M_1 + M_2)$. This equation is that of the motion of M_2 referred to O_1 , the center of mass of M_1 , as origin.

Equations (3.7) reduce to Keplerian motion when the two masses M_1 and M_2 are spheres which are homogeneous in concentric layers. For in this case the moments of inertia about the principal axes are equal and

$$2J_q = A_q + B_q + C_q = 3A_q,$$

$$(R \cdot \hat{\Phi}_q \cdot R) / r^2 = [R \cdot (A_q i_q i_q + B_q j_q j_q + C_q k_q k_q) \cdot R] / r^2 = A_q,$$

and

$$R \cdot \hat{\Phi}_q = A_q R \cdot (i_q i_q + j_q j_q + k_q k_q) = A_q R, \quad q = 1, 2;$$

where J_q , $\hat{\Phi}_q$, A_q , B_q , and C_q have the same interpretation as given in the previous section. On substituting these values in equation (3.8) it becomes

$$R'' = -[k^2(M_1 + M_2)/r^3]R,$$

which is the equation of relative motion for homogeneous spheres.

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If the bodies are nearly spheres so that the Keplerian motion can be taken as a first approximation, then the perturbative function can be written as

$$(3.9) \quad P = k^2 \left[-(3M_2 J_1 / r^5) R + (15M_2 / 2r^7) R \cdot \hat{\Phi}_1 \cdot RR - (3M_2 / r^5) R \cdot \hat{\Phi}_1 \right. \\ \left. - (3M_1 J_2 / r^5) R + (15M_1 / 2r^7) R \cdot \hat{\Phi}_2 \cdot RR - (3M_1 / r^5) R \cdot \hat{\Phi}_2 \right].$$

To evaluate this function not only must the relative positions of the bodies be known but also the orientations of their principal axes.

4. Differential Equations for Rotation of Bodies.—The equations of motion of the body M_p about a fixed point O_p is in dyadic notation⁷

$$(4.1) \quad (\hat{\Phi}_p \cdot W_p)' = \hat{\Phi}_p \cdot W_p' + W_p \times \hat{\Phi}_p \cdot W_p = N_p, \quad p = 1, 2,$$

where W_p is the angular velocity, $\hat{\Phi}_p \cdot W_p$ the angular momentum about O_p , and N_p is the moment of the exterior forces that are acting on M_p ($p = 1, 2$). The dyadic $\hat{\Phi}_1$ depends upon the distribution of matter about the point O_1 fixed in the body M_1 and is constant relative to a set of axes fixed in the body with origin at O_1 .

When the vectors in equation (4.1) are expressed in terms of their rectangular components parallel to the principal axes of the body at O_1 (4.1) becomes the dynamical equations of Euler.⁸

The value of the moment of the exterior forces N_1 required in equation (4.1) is obtainable from the potential function. Since the work done in an infinitesimal rotation of a body M_1 is equal to the corresponding change in the potential function

$$(4.2) \quad N_1 \cdot d\theta_1 = dV_1$$

where the vector $d\theta_1$ is an infinitesimal rotation about an axis through the center of mass, and N_1 the moment of the exterior forces about the center of mass acting on the body M_1 . It follows from (2.8) that the change, dV_1 , in the potential function for an infinitesimal rotation $d\theta_1$ of the body M_1 is

⁷Weatherburn, op. cit., p. 110.

⁸W. D. MacMillan, Dynamics of Rigid Bodies (New York: McGraw-Hill Co., Inc., 1936), p. 184.

$$(4.3) \quad dV_1 = -(3k^2 M_2 / 2r^5) d(R \cdot \bar{\Phi}_1 \cdot R) = -(3k^2 M_2 / 2r^5) R \cdot d\bar{\Phi}_1 \cdot R$$

since R remains constant and only $\bar{\Phi}_1$ changes. Now in an infinitesimal rotation $d\theta_1$ the change in the unit vectors i_1, j_1, k_1 , are respectively

$$d\theta_1 \times i_1, d\theta_1 \times j_1, d\theta_1 \times k_1$$

so that

$$\begin{aligned} d\bar{\Phi}_1 &= A_1 i_1 di_1 + B_1 j_1 dj_1 + C_1 k_1 dk_1 + A_1 di_1 i_1 + B_1 dj_1 j_1 + C_1 dk_1 k_1 \\ &= -\bar{\Phi}_1 \times d\theta_1 + d\theta_1 \times \bar{\Phi}_1, \end{aligned}$$

and therefore

$$\begin{aligned} (4.4) \quad dV_1 &= -(3k^2 M_2 / 2r^5) R \cdot (-\bar{\Phi}_1 \times d\theta_1 + d\theta_1 \times \bar{\Phi}_1) \cdot R \\ &= (3k^2 M_2 / r^5) (R \cdot \bar{\Phi}_1 \times R) \cdot d\theta_1. \end{aligned}$$

On comparing this result with (4.2) it follows that

$$(4.5) \quad N_1 = (3k^2 M_2 / r^5) (R \times \bar{\Phi}_1 \cdot R).$$

In like manner for an infinitesimal rotation of M_2 we have

$$(4.6) \quad N_2 = (3k^2 M_1 / r^5) (R \times \bar{\Phi}_2 \cdot R).$$

The equation of motion (4.1) of the body M_p about the fixed point O_p in M_p is therefore

$$\begin{aligned} (4.7) \quad \bar{\Phi}_p \cdot W'_p + W_p \times \bar{\Phi}_p \cdot W_p &= (3k^2 M_q / r^5) (R \times \bar{\Phi}_p \cdot R), \\ p &= 1, 2, \quad q = 2, 1. \end{aligned}$$

When expressed in terms of components along principal axes (4.7) becomes, for case $p = 1, q = 2$,

$$\begin{aligned} (4.8) \quad A_1 W_{i_1} + (C_1 - B_1) W_{j_1} W_{k_1} &= 3k^2 M_2 (C_1 - B_1) Y_1 Z_1 / r^5 \\ B_1 W_{j_1} + (A_1 - C_1) W_{k_1} W_{i_1} &= 3k^2 M_2 (A_1 - C_1) Z_1 X_1 / r^5 \\ C_1 W_{k_1} + (B_1 - A_1) W_{i_1} W_{j_1} &= 3k^2 M_2 (B_1 - A_1) X_1 Y_1 / r^5 \end{aligned}$$

where X_1, Y_1, Z_1 are the co-ordinates of O_2 referred to the principal axes of M_1 and $W_{i_1}, W_{j_1}, W_{k_1}$ are the components of the angular velocity W_1 on these axes.

There is a similar set of equations for the body M_2 ($p = 2, q = 1$).

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It might be worthy of note that, if the mass M_1 is taken as the earth and M_2 as the mass of the sun, the equations (4.8) give those obtained by Tisserand⁹ after a similar set of external moments have been added for the effect of the moon. The amount of work in the preceding derivation of equations (4.8) being much less than their derivation in the Traité De Mécanique Céleste.

The equations of motion of a system of two rigid distinct bodies M_1 and M_2 , acting under their mutual attraction and free from all external forces, when referred to a set of fixed axes in space, are of the twenty-fourth order. Here the equations of motion are referred to the mass center O_1 of the body M_1 as origin and the equations of motion reduce to a system of the eighteenth order that are given by equations (3.8) and (4.7) together with the equations

$$(4.9) \quad \begin{aligned} W_{1p} &= \psi p' \sin \theta_p \sin \phi_p + \theta_p' \cos \phi_p, \\ W_{2p} &= \psi p' \sin \theta_p \cos \phi_p - \theta_p' \sin \phi_p, \\ W_{3p} &= \psi p' \cos \theta_p - \phi_p', \quad p = 1, 2, \end{aligned}$$

where ψ , θ , ϕ , ψ' , θ' , and ϕ' are Euler's angles¹⁰ and their derivatives with respect to the time and the W 's are the components of the angular velocity of the body.

In these equations we have neglected fourth and higher powers of $1/r$. With this approximation the motion of any two rigid bodies under their mutual gravitation is the same as that of two ellipsoids with equivalent moments of inertia.

5. Energy Integral.—The entire kinetic energy of the system is the kinetic energies of rotation of the bodies M_1 and M_2 , plus the kinetic energies of revolution of M_1 and M_2 about their center of mass O . Thus

$$(5.1) \quad K.E. = [MR'^2 + W_1 \cdot \phi_1 \cdot W_1 + W_2 \cdot \phi_2 \cdot W_2] / 2.$$

The energy integral will now be obtained by showing that the derivative of the potential function is equal to the derivative

⁹ F. Tisserand, Traité De Mécanique Céleste, Tome II (Paris: Gauthier-Villars et Fils, Imprimeurs-Libraires Des Longitudes, De L'Ecole Polytechnique, 1891), p. 372.

¹⁰ MacMillan, Dynamics of Rigid Bodies, p. 185.

of the kinetic energy.

The derivative of the potential function with respect to the time is

$$\begin{aligned}
 (5.2) \quad V' = & k^2 \{ -(M_1 M_2 / r^3) R \cdot R' - (3M_2 J_1 / r^5) R \cdot R' \\
 & + (15M_2 / 2r^7) R \cdot \Phi_1 \cdot R R \cdot R' - (3M_2 / r^5) R \cdot \Phi \cdot R' \\
 & - (3M_2 / r^5) R \cdot W_1 \times \Phi_1 \cdot R - (3M_1 J_2 / r^5) R \cdot R' \\
 & + (15M_1 / 2r^7) R \cdot \Phi_2 \cdot R R \cdot R' - (3M_1 / r^5) R \cdot \Phi_2 \cdot R' \\
 & - (3M_1 / r^5) R \cdot W_2 \times \Phi_2 \cdot R + \dots \}.
 \end{aligned}$$

The derivative of the kinetic energy is equal to one-half the derivative of

$$MR'^2$$

plus one-half the derivative of

$$W_1 \cdot \Phi_1 \cdot W_1 + W_2 \cdot \Phi_2 \cdot W_2.$$

One-half the derivative with respect to the time of the former is

$$(5.3) \quad MR' \cdot R'' = R' \cdot \nabla V,$$

where

$$\begin{aligned}
 R' \cdot \nabla V = & k^2 \{ -(M_1 M_2 / r^3) R \cdot R' - (3M_2 J_1 / r^5) R \cdot R' \\
 & + (15M_2 / 2r^7) R \cdot \Phi_1 \cdot R R \cdot R' - (3M_2 / r^5) R \cdot \Phi \cdot R' \\
 & - (3M_1 J_2 / r^5) R \cdot R' + (15M_1 / 2r^7) R \cdot \Phi_2 \cdot R R \cdot R' \\
 & - (3M_1 / r^5) R \cdot \Phi_2 \cdot R' + \dots \},
 \end{aligned}$$

as is seen by taking the scalar product of (3.4) with R' .

Since Φ_p is a self-conjugate dyadic,

$$\begin{aligned}
 (5.4) \quad (W_p \cdot \Phi_p \cdot W_p)' &= 2(W_p \cdot \Phi_p \cdot W'_p + W_p \cdot W_p \times \Phi_p \cdot W_p) \\
 &= 2W_p \cdot \Phi_p \cdot W'_p.
 \end{aligned}$$

It is seen by taking the scalar product of (4.1) with W_p , after substituting the value obtained in section 4 for N_p , that

$$W_p \cdot (\Phi_p \cdot W_p)' = W_p \cdot \Phi_p \cdot W'_p = (3k^2 M_q / r^5) W_p \cdot R \times \Phi_p \cdot R,$$

and therefore, by (5.4),

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$$(5.5) \quad (\mathbf{W}_p \cdot \dot{\mathbf{f}}_p \cdot \mathbf{W}_p)' = (6k^2 M_q / r^5) \mathbf{W}_p \cdot \mathbf{R} \times \dot{\mathbf{f}}_p \cdot \mathbf{R} \\ = -(6k^2 M_q / r^5) \mathbf{R} \cdot \mathbf{W}_p \times \dot{\mathbf{f}}_p \cdot \mathbf{R}, \quad p = 1, 2, \\ q = 2, 1.$$

On comparing the foregoing results, it is seen that

$$(5.6) \quad M(R')^2 + (\mathbf{W}_1 \cdot \dot{\mathbf{f}}_1 \cdot \mathbf{W}_1)' + (\mathbf{W}_2 \cdot \dot{\mathbf{f}}_2 \cdot \mathbf{W}_2)' = 2V'$$

which is a perfect differential equation. On integrating this expression we obtain the energy integral

$$(5.7) \quad (MR')^2 + \mathbf{W}_1 \cdot \dot{\mathbf{f}}_1 \cdot \mathbf{W}_1 + \mathbf{W}_2 \cdot \dot{\mathbf{f}}_2 \cdot \mathbf{W}_2) / 2 \\ = k^2 [M_1 M_2 / r + M_2 J_1 / r^3 - (3M_2 / 2r^5) \mathbf{R} \cdot \dot{\mathbf{f}}_1 \cdot \mathbf{R} + M_1 J_2 / r^3 \\ - 3M_1 / 2r^5) \mathbf{R} \cdot \dot{\mathbf{f}}_2 \cdot \mathbf{R}] + c$$

where c is the energy constant.

6. Moment of Momentum Integral.—The moment of momentum of the system here considered, since there are no external forces, is a constant. This result can be established directly as follows.

The rate of change of the angular momentum of the body M_p is by equations (4.1) and (4.7)

$$(6.1) \quad (\dot{\mathbf{f}}_p \cdot \mathbf{W}_p)' = (3k^2 M_q / r^5) \mathbf{R} \times \dot{\mathbf{f}}_p \cdot \mathbf{R}, \quad p = 1, 2, \\ q = 2, 1.$$

Equation (3.8) gives

$$(6.2) \quad MR'' = k^2 [(-M_1 M_2 / r^3) \mathbf{R} - (M_2 J_1 / r^5) \mathbf{R} + (15M_2 / 2r^7) \mathbf{R} \cdot \dot{\mathbf{f}}_1 \cdot \mathbf{R} \\ - (3M_2 / r^5) \mathbf{R} \cdot \dot{\mathbf{f}}_1 - M_1 J_2 / r^5 + (15M_1 / 2r^7) \mathbf{R} \cdot \dot{\mathbf{f}}_2 \cdot \mathbf{R} \\ - (3M_1 / r^3) \mathbf{R} \cdot \dot{\mathbf{f}}_2],$$

from which it is seen that

$$(6.3) \quad MR \times R'' = -k^2 [(3M_2 / r^5) \mathbf{R} \times \dot{\mathbf{f}}_1 \cdot \mathbf{R} + (3M_1 / r^5) \mathbf{R} \times \dot{\mathbf{f}}_2 \cdot \mathbf{R}].$$

It follows from the identity

$$(6.4) \quad \mathbf{R} \times \mathbf{R}'' = (\mathbf{R} \times \mathbf{R}')',$$

and equations (6.1) and (6.3), that

$$(6.5) \quad \sum_{p=1}^2 (\dot{\mathbf{f}}_p \cdot \mathbf{W}_p)' + M(\mathbf{R} \times \mathbf{R}')' = 0.$$

Integrating equation (6.5) gives the moment of momentum integral

$$(6.6) \quad \dot{\mathbf{f}}_1 \cdot \mathbf{W}_1 + \dot{\mathbf{f}}_2 \cdot \mathbf{W}_2 + MR \times \mathbf{R}' = D.$$

The vector D gives the direction of the invariable axis of the system which is determined if the initial conditions are known. It also gives the magnitude of the moment of momentum and is therefore equivalent to three arbitrary scalar constants.

The energy and moment of momentum integrals would enable us to reduce the differential equations of the system from one of the eighteenth order, sec. 4, to one of the fourteenth order. No other algebraic integrals were discovered in the general case of two rigid bodies.

7. Differential Equations of Motion of Spheroid and Sphere.—We shall now consider the special case of a spheroid and a sphere. Let the spheroid be denoted by M_1 and the sphere by M_2 . The axis in M_1 is taken as the principal axis of inertia. The axes of reference for the system with fixed directions are taken with origin at the center of mass of M_1 at O_1 in the spheroid with the z -axis having the same direction as the vector constant of the moment of momentum integral.

Since the moment of inertia A_1 is equal to B_1 for a spheroid, the inertia dyadic $\hat{\phi}_1$ for M_1 , as defined in section 2, can be written

$$\hat{\phi}_1 = A_1 I + (C_1 - A_1) k_1 k_1$$

where I is the identity dyadic

$$I = i_1 i_1 + j_1 j_1 + k_1 k_1 = i_2 i_2 + j_2 j_2 + k_2 k_2$$

and, since for the case of the sphere $A_1 = B_1 = C_1$, the value of $\hat{\phi}_2$ reduces to

$$\hat{\phi}_2 = A_2 I.$$

For the spheroid M_1 equation (4.7) becomes

$$\begin{aligned} (7.1) \quad A_1 W'_1 + (C_1 - A_1)(k_1 \cdot W'_1) k_1 + (C_1 - A_1)(W_1 \times k_1)(k_1 \cdot W_1) \\ = 3(C_1 - A_1)(k^2 M_2 / r^5)(R \times k_1)(k_1 \cdot R). \end{aligned}$$

On taking the scalar product of (7.1) with k_1 we have

$$(7.2) \quad C_1 k_1 \cdot W'_1 = 0.$$

Since

$$(7.3) \quad k'_1 = W_1 \times k_1$$

it follows that

$$(7.4) \quad k'_1 \cdot W_1 = 0,$$

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and therefore from equations (7.2) and (7.3)

$$(7.5) \quad C_1(k_1 \cdot W_1)' = C_1 k_1 \cdot W_1' + C_1 k_1' \cdot W_1 = 0.$$

On integrating equations (7.5) we have

$$(7.6) \quad k_1 \cdot W_1 = b$$

where b is a constant of integration. Equation (7.6) says that the projections of the angular velocity on the k_1 axis of the spheroid is a constant.

For the sphere M_2 equation (4.7) reduces to

$$A_2 W_2' = 0.$$

On integrating this equation

$$(7.7) \quad A_2 W_2 = D_2.$$

Since this is a vector equation, it is equivalent to three integrals and indicates that the angular velocity of a sphere is constant when acted upon only by the attraction of an exterior body, which one would expect when only gravitational forces are acting on a sphere.

The moment of momentum integral, equation (6.6), is in this case

$$(7.8) \quad A_1 W_1 + (C_1 - A_1) b k_1 + M R \times R' = D,$$

where D is a constant vector. By letting

$$(7.9) \quad S = M R \times R' - D,$$

equation (7.8) becomes

$$(7.10) \quad A_1 W_1 = -(C_1 - A_1) b k_1 - S$$

and hence by (7.6)

$$(7.11) \quad S \cdot k_1 = -C_1 b.$$

The energy integral, equation (5.7), takes the form

$$(7.12) \quad M R'^2 + A_1 W_1^2 = 2k^2 M_1 M_2 / r + (C_1 - A_1) k^2 M_2 / r^3 \\ - (C_1 - A_1) (3k^2 M_2 / r^3) (R \cdot k_1)^2 + a$$

where

$$a = -(C_1 - A_1) b^2 - A_2 W_2^2 + c$$

and c is equal to twice the constant of integration of equation (5.7).

The differential equation for relative motion, equation (3.8), becomes

$$\begin{aligned}
 (7.13) \quad MR'' &= k^2 \left\{ -M_1 M_2 / r^3 - 3(C_1 - A_1) M_2 / 2r^5 \right. \\
 &\quad \left. + [15(C_1 - A_1) M_2 / 2r^7] (R \cdot k_1)^2 \right\} R \\
 &\quad - [3(C_1 - A_1) k^2 M_2 / r^5] (R \cdot k_1) k_1.
 \end{aligned}$$

By substituting the values found in equations (7.2), (7.3), and (7.6) in equation (7.1), the latter takes the form

$$(7.14) \quad A_1 W' + (C_1 - A_1) b k_1' = [3(C_1 - A_1) k^2 M_2 / r^5] (R \times k_1) (k_1 \cdot R).$$

On taking the vector product of (7.13) with R the result is

$$(7.15) \quad MR \times R'' = -[3(C_1 - A_1) k^2 M_2 / r^5] (R \cdot k_1) (k_1 \times R).$$

On substituting the value of W_1 from equation (7.8) in equation (7.12) and solving for $(R \cdot k_1)^2$ there results the equation

$$(7.16) \quad (R \cdot k_1)^2 = \frac{2k^2 M_1 M_2 / r + (C_1 - A_1) k^2 M_2 / r^3 - MR'^2 - S^2 / A_1 + h}{3(C_1 - A_1) k^2 M_2 / r^5},$$

where the constant

$$h = (C_1 - A_1) b^2 (C_1 / A_1) - A_2 W_2^2 + c.$$

It will be noted that in the case of a spheroid and a sphere a vector integral and a scalar integral were readily obtainable and these additional integrals could be used to reduce the problem from one of the fourteenth order, sec. 6, to one of the tenth order.

8. Motion of Spheroid and Particle.—A limiting case of sec. 7 is that of the motion of a spheroid and a particle. For this case M_2 approaches zero and equation (7.8) becomes

$$(8.1) \quad A_1 W_1 + (C_1 - A_1) b k_1 = D.$$

On taking the vector product of (8.1) with k_1 and applying (7.3) it follows that

$$(8.2) \quad k_1' = W \times k_1$$

where

$$W = D / A_1$$

and hence is a constant vector. That is, the axis of the spheroid describes a cone with a constant angular velocity, a result that is well known.¹¹

¹¹ MacMillan, Dynamics of Rigid Bodies, chap. vii.

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Then again by letting M_2 approach zero in (7.13),

$$(8.3) \quad R'' = k^2 [(-M_1/r^3 - \{3(C_1 - A_1)/2r^5\} + \{15(C_1 - A_1)/2r^7\} \\ [R \cdot k_1]^2)R - \{3(C_1 - A_1)/r^5\}(R \cdot k_1)k_1].$$

By taking the vector product of (8.3) with R and forming the scalar product of the result with W

$$(8.4) \quad W \cdot (R \times R')' = -\{3k^2(C_1 - A_1)/r^5\}(R \cdot k_1)[W \cdot (R \times k_1)], \\ = \{3k^2(C_1 - A_1)/r^5\}(R \cdot k_1)(R \cdot k_1'),$$

the last equality being a consequence of (8.2).

On taking the scalar product of (8.3) with R'

$$(8.5) \quad R' \cdot R'' = k^2 [(-M_1/r^3 - 3[C_1 - A_1]/2r^5 \\ + \{15(C_1 - A_1)/2r^7\}[R \cdot k_1]^2)R' \cdot R \\ - \{3(C_1 - A_1)/r^5\}(R \cdot k_1)(R' \cdot k_1)].$$

Or, by (8.4), equation (8.5) can be expressed in the form

$$(8.6) \quad (R'^2)' = 2k^2 [(\frac{M_1}{r} + \frac{[C_1 - A_1]}{2r^3})' - (\frac{3[C_1 - A_1]}{2r^5} [R \cdot k_1]^2)' \\ + (W \cdot [R \times R'])'],$$

which on integrating gives

$$(8.7) \quad R'^2 - 2W \cdot (R \times R') = 2k^2 M_1/r + [k^2(C_1 - A_1)/r^3] [1 - \frac{3(R \cdot k_1)^2}{r^2}] \\ + \text{constant}.$$

If i_0, j_0, k_0 is any orthogonal system rotating with constant angular velocity W and x, y, z are the co-ordinates of R in this system, then the actual velocity¹²

$$R' = v + W \times R, \quad \rho = R,$$

and hence the square of the velocity, v , relative to this system is

$$(8.8) \quad v^2 = (R' - W \times R)^2 \\ = R'^2 - 2(W \times R) \cdot R' + W^2 R^2 - (W \cdot R)^2$$

where v is the relative velocity in the i_0, j_0, k_0 system.

On substituting this result in equation (8.7)

¹²Ibid., p. 176.

$$(8.9) \quad v^2 = 2k^2 M_1/r + k^2 (C_1 - A_1) (1 - (3/r^2) [R \cdot k_1]^2) / r^3 + W^2 R^2 - (W \cdot R)^2 + \text{constant}.$$

This integral resembles Jacobi's integral in the three-body problem.¹³ It is a relation between the square of the velocity relative to the rotating axes in the spheroid and the coordinates of the infinitesimal body revolving about it. In arriving at this result it was assumed that higher terms in the expansion of the potential could be neglected. It is, however, not necessary to make this restriction in order to arrive at this integral. In fact it can be shown that

$$(8.10) \quad v^2 = 2V(R) + W^2 R^2 - (W \cdot R)^2 + \text{constant},$$

where $V(R)$ is the potential of a spheroid with axis k_1 . When the constants have been determined by the initial conditions the integral determines the velocity with which the infinitesimal body will move, if at all, at all points with reference to the axes in the spheroid; and conversely, for a given velocity, it gives the locus of those points of relative space where alone the infinitesimal body can be. In particular, if v is put equal to zero the integral will define the surfaces at which the relative velocity must be zero provided of course the body reaches the surface. On one side of these surfaces the velocity will be real and on the other side imaginary and therefore it shows in what portions of relative space the infinitesimal body can move and in what portions it can not.

The only requirement on the i_0 , j_0 , k_0 was that it rotate with constant velocity W . Hence let us assume that the unit vector k_0 is along W and that i_0 is in the plane of the W and the axis k_1 of the spheroid. Then

$$k_1 = i_0 \cos \alpha + k_0 \sin \alpha$$

where α is the constant angle between k_1 and i_0 . With this notation equation (8.9) becomes

$$(8.11) \quad v^2 = 2k^2 M_1/r + k^2 (C_1 - A_1) [1 - (3/r^2) (x \cos \alpha + z \sin \alpha)^2] / r^3 + W^2 (x^2 + y^2).$$

It is convenient to choose the unit of mass as M_1 , and the

¹³F. R. Moulton, An Introduction to Celestial Mechanics (New York: Macmillan Co., 1923), p. 281.

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units of distance and time such that k^2 and W^2 are both equal to unity. With this notation the surfaces of zero velocity becomes

$$(8.12) \quad x^2 + y^2 + \frac{2}{r} \left[1 + \frac{(C_1 - A_1)}{2r^2} \left(1 - \frac{3[x \cos \alpha + z \sin \alpha]^2}{r^2} \right) \right] = C,$$

$$r^2 = x^2 + y^2 + z^2.$$

If the eccentricity of the spheroid becomes a sphere (8.12) reduces to

$$(8.13) \quad x^2 + y^2 + 2/r = C,$$

and it is seen that the difference between this limiting case and equation (8.12) is actually twice the difference between the potential of a sphere and the potential of a spheroid of the same mass. If the eccentricity of the spheroid is small equation (8.13) gives an approximation to its surfaces of zero relative velocity. In Figs. 2 and 3 the traces of the surfaces (8.13) on the zx -plane and the xy -plane for different values of the constant C are plotted and indicated by the broken lines. On the same graph are indicated by the heavy lines the changes that take place when a small value is assigned to the difference in inertia ($C_1 - A_1$). The angle $k_1 - 0 - t$ in Fig. 2 is equal to the arc $\cos V\sqrt{3}/3$ and Fig. 3 represents the traces in the xy -plane when the k_1 -axis is such that it makes an angle less than the arc $\cos V\sqrt{3}/3$ with the x -axis, otherwise the continuous curves would not intersect the broken ones (Fig. 3, curves C_1, C_2).

It follows from the general forms of the surfaces that the double points which appear as C is given different values are all in the xy -plane; and that, with the exception of the case when the k_1 axis is the w -axis, these double points will lie on the x - and y -axes in pairs, the two points in each pair being equally distant from the origin due to the symmetry of the traces in the xy -plane. The pair on the x -axis appear as the surfaces vanish from the xy -plane from above and below. The other pair, on the y -axis, appear as the spheroidal shaped surfaces on the interior are separating from the cylindrical shaped surfaces on the exterior.

Since from (8.10)

$$v^2 = 2V(R) + (W^2 R^2 - [W \cdot R]^2) + \text{constant}.$$

It follows that

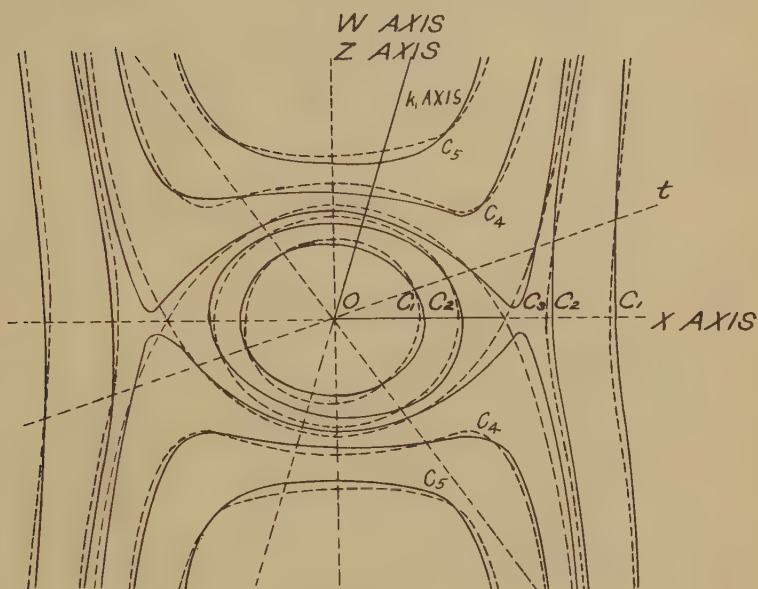


FIG. 2.

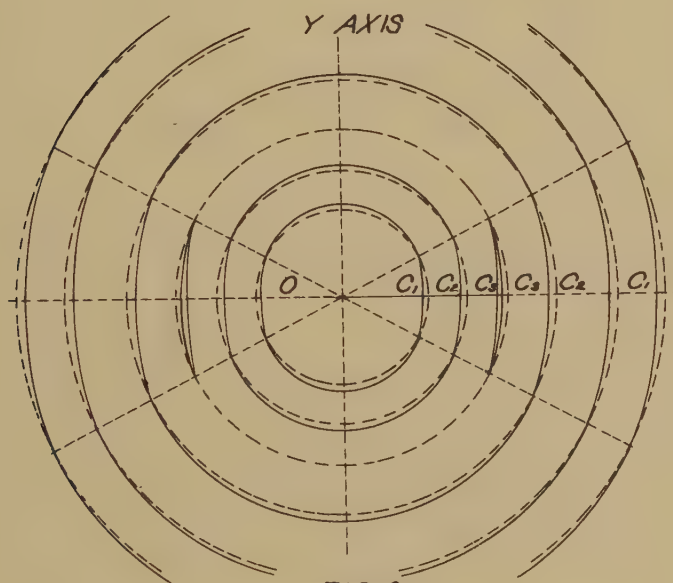


FIG. 3.

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$$\begin{aligned}(\nabla v^2)/2 &= \nabla V(R) + \nabla [W^2 R^2 - (W \cdot R)^2]/2, \\ &= R'' + W^2 R - (W \cdot R)W,\end{aligned}$$

where W is considered as a constant in direction and magnitude; and since, when the co-ordinates of the particle are referred to the rotating system,

$$R'' = \rho'' + 2W \times \rho' + (W' \times \rho + [W \cdot \rho]W - W^2 \rho),$$

where the vector $\rho = R$, $\rho' = v$, and ρ'' is the relative acceleration, it is seen that

$$\nabla v^2/2 = \rho'' + 2W \times v.$$

This indicates that on the surfaces of zero relative velocity the direction of acceleration, or the lines of effective force, are orthogonal to these surfaces. If then for double points on the surfaces of zero relative velocity

$$v^2/2 = 0 = \rho'',$$

it follows that an infinitesimal body being placed at a double point with zero relative velocity will remain there indefinitely under the conditions of this problem, since then its co-ordinates will identically fulfill the differential equations of motion. This implies that such a particle will actually travel in a circle about the W -axis.

Having determined the forms of the surfaces, it remains to find in what region of relative space the motion is real and in what regions it is imaginary. For large values of C the ovals and curtains are separated and it is seen from equation (8.11) that x and y can be taken large enough so that v^2 will be positive. Then for large values of C the motion will be real outside the surface; imaginary between these outer folds and the ovals corresponding to them; and real inside the ovals (Fig. 2, curves C_1, C_2). If then the particle has a C value equal to say C_1 and is outside the cylindrical shaped surface C_1 it will remain outside and if it is inside the spheroidal shaped surface C_1 it will remain having no way of escape. As C becomes smaller the outside surfaces approach the inside surfaces finally meeting between curves C_2 and C_4 (Fig. 2). As C is made still smaller the motion is real within the over and under hanging folds or curtains (Fig. 2, curves C_4, C_5).

When the principal axis of M_1 is taken as one of the fixed axes of reference equation (8.3) reduces to

$$(8.14) \quad R'' = k^2 [(-M_1/r^3 - 3M_1 a_1^2 e_1^2 / 10r^5 + 15M_1 a_1^2 e_1^2 z^2 / 10r^7) R - 3M_1 a_1^2 e_1^2 z k_1 / 5r^5],$$

since for a spheroid

$$A_1 = M_1 (a_1^2 + b_1^2) / 5,$$

$$C_1 = 2M_1 a_1^2 / 5,$$

$$C_1 - A_1 = M_1 (a_1^2 - b_1^2) / 5,$$

$$a_1^2 - b_1^2 = a_1^2 e_1^2,$$

where a and b are the semi-major and minor axes of the ellipsoid respectively, and e is the eccentricity of the meridian section. Equation (8.14) can be expressed in its rectangular components as

$$x'' = -k^2 M_1 x [1 + 3a_1^2 e_1^2 (x^2 + y^2 - 4z^2) / 10r^4 \dots] / r^3,$$

$$y'' = -k^2 M_1 y [1 + 3a_1^2 e_1^2 (x^2 + y^2 - 4z^2) / 10r^4 \dots] / r^3,$$

$$z'' = -k^2 M_1 z [1 + 3a_1^2 e_1^2 (3x^2 + 3y^2 - 2z^2) / 10r^4 \dots] / r^3.$$

The projection of the areal velocity on the xy -plane is seen to be

$$xy' - yx' = c_1.$$

The vis viva integral is

$$x'^2 + y'^2 + z'^2 = 2v + c_2.$$

These equations are treated by W. D. Macmillan,¹⁴ where in the MacMillan case the axis is fixed in the integral investigated in this section.

9. Removal of Variables Associated with Rotation.—We shall now return to the more general situation above of a spheroid and a sphere, sec. 7, and, by resolving the vector k_1 into three chosen orthogonal directions, express the vector R'' in terms of R and R' thus removing the variables associated with rotation. To accomplish this reduction it is, by (7.13), necessary to express the axis k_1 of the spheroid in terms of R and R' . Now since the

¹⁴F. R. Moulton, Periodic Orbits, chap. iv, p. 99, "Periodic Orbits About an Oblate Spheroid."

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vector S , by (7.9), is a function of R and R' alone it will be convenient to choose the directions of the vectors R , $R \times S$, and $R \times (R \times S)$ for this resolution. We have therefore

$$(9.1) \quad k_1 = \xi R + \eta R \times S + \zeta R \times (R \times S).$$

Then

$$(9.2) \quad R \cdot k_1 = \xi R^2,$$

$$(9.3) \quad R \times (R \times S) \cdot k_1 = \xi R^2 (R \times S)^2,$$

and

$$(9.4) \quad k_1^2 = 1 = \xi^2 R^2 + \eta^2 (R \times S)^2 + \zeta^2 R^2 (R \times S)^2.$$

From equations (9.1), (9.2), and (9.3), it follows that

$$(9.5) \quad (R \times S)^2 = \pm \sqrt{(S \times k_1)^2 (R \times k_1)^2 - [(R \cdot k_1)(S \cdot k_1) - (R \cdot S)]^2}.$$

On substituting the values of ξ , η , and ζ obtained from equations (9.2), (9.3), and (9.5) in equation (9.1) it is found that

$$(9.6) \quad k_1 = \frac{R \cdot k_1}{R^2} R + R \times (R \times S) \cdot k_1 \frac{R \times (R \times S)}{[R \times (R \times S)]^2} \\ \pm [(S \times k_1)^2 (R \times k_1)^2 - (R \cdot k_1 S \cdot k_1 - R \cdot S)^2]^{\frac{1}{2}} \frac{R \times S}{(R \times S)^2},$$

the right member of which is expressible in terms of R and R' only; since, by (7.11), $S \cdot k_1$ is a constant; by (7.16), $R \cdot k_1$ is a function of R , R' and S ; and, as stated above, S is a function of R and R' .

We are now in a position to express R'' as a function of R and R' . It will be convenient, however, to still retain the symbol S remembering that it is expressible in R and R' and given by (7.9). These substitutions will be made as follows. From equation (9.6)

$$(9.7) \quad k_1 = \frac{R \cdot k_1}{R^2} R - (R \cdot D R \cdot k_1 - R^2 C_1 b) \frac{R \times (R \times S)}{[R \times (R \times S)]^2} \pm [S^2 R^2 \\ - S^2 (R \cdot k_1)^2 - R^2 (C_1 b)^2 + 2 C_1 b R \cdot D R \cdot k_1 - (R \cdot D)^2]^{\frac{1}{2}} \frac{R \times S}{(R \times S)^2}$$

after substitutions (7.11),

$$S \cdot k_1 = -C_1 b,$$

and (7.9),

$$R \cdot S = -R \cdot D,$$

have been made. Then (7.13) becomes

$$\begin{aligned}
 (9.8) \quad MR'' = & \left[\frac{-k^2 M_1 M_2}{r^3} - \frac{3k^2 M_2 (C_1 - A_1)}{2r^5} + \frac{9k^2 M_2 (C_1 - A_1)}{2r^7} (R \cdot k_1)^2 \right] R \\
 & + \frac{3k^2 M_2 (C_1 - A_1)}{r^5} [R \cdot D(R \cdot k_1)^2 - R^2 C_1 b R \cdot k_1] \frac{R \times (R \times S)}{[R \times (R \times S)]^2} \\
 & + \frac{3k^2 M_2 (C_1 - A_1) R \cdot k_1}{r^5} [S^2 R^2 - S^2 (R \cdot k_1)^2 - R^2 (C_1 b)^2 \\
 & + 2C_1 b R \cdot D R \cdot k_1 - (R \cdot D)^2] \frac{1}{2} \frac{R \times S}{(R \times S)^2}.
 \end{aligned}$$

On substituting in the expression above

$$(9.9) \quad R \cdot k_1 = \pm \frac{r^5}{3M_2 k^2 (C_1 - A_1)} \left[\frac{3M_2 k^2 (C_1 - A_1)}{r^5} \left(\frac{k^2 M_2 (C_1 - A_1)}{r^3} + K \right) \right]^{\frac{1}{2}}$$

from (7.16), where

$$K = 2k^2 M_1 M_2 / r - MR'^2 - S^2 / A_1 + h,$$

we obtain the desired equation

$$(9.10) \quad MR'' = - \frac{k^2 M_1 M_2}{r^3} R + Q \frac{R}{r} + G \frac{R \times (R \times S)}{\sqrt{[R \times (R \times S)]^2}} + H \frac{R \times S}{\sqrt{(R \times S)^2}},$$

where

$$(9.11) \quad Q = (3/2r)K,$$

$$\begin{aligned}
 (9.12) \quad G = & \left\{ \left[K + \frac{k^2 M_2 (C_1 - A_1)}{r^3} \right] R \cdot D + \left[\frac{3k^2 M_2 k^2 (C_1 - A_1)}{r^5} \left(\frac{k^2 M_2 (C_1 - A_1)}{r^3} \right. \right. \right. \\
 & \left. \left. + K \right) \right]^{\frac{1}{2}} R^2 C_1 b \right\} \frac{\sqrt{[R \times (R \times S)]^2}}{[R \times (R \times S)]^2},
 \end{aligned}$$

and

$$\begin{aligned}
 (9.13) \quad H = & \left\{ \left[K + \frac{k^2 M_2 (C_1 - A_1)}{r^3} \right] \frac{k^2 M_2 (C_1 - A_1)}{r^3} (2S^2 - 3C_1^2 b^2) - KS^2 \right. \\
 & + \frac{2C_1 b R \cdot D}{r} \left[\frac{3k^2 M_2 (C_1 - A_1)}{r^3} \left(\frac{k^2 M_2 (C_1 - A_1)}{r^3} + K \right) \right]^{\frac{1}{2}} \\
 & \left. - \frac{3k^2 M_2 (C_1 - A_1) (R \cdot D)^2}{r^5} \right\} \frac{1}{2} \frac{\sqrt{(R \times S)^2}}{(R \times S)^2}.
 \end{aligned}$$

This sixth order system involves only R and its first and second derivatives with respect to the time as variables. The

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 acceleration, R'' , is here resolved along the radius vector and two conveniently chosen directions at right angles to the radius vector and to each other. The expression for K approaches zero when $C_1 \rightarrow A_1$, as can be seen from equation (9.9) by multiplying both sides by $(C_1 - A_1)$ and letting $C_1 \rightarrow A_1$ as a limit. The equation (9.10), therefore, becomes for a sphere

$$(9.14) \quad MR'' = -(k^2 M_1 M_2 / r^3) R,$$

which is the equation of the ordinary two-body problem in vector notation.

In the case of the ordinary two-body problem, the body M_2 will move in a conic section whose elements can be uniquely determined.¹⁵ If $(C_1 - A_1)$ is not too large we can think of the terms in the right member of equation (9.10), with the exception of the first, as three rectangular components of disturbing forces which will continually change the elements of the orbit which the body would have if it were moving undisturbed in a conic section. The differences between the co-ordinates and the components of velocities in the actual orbits and those which the bodies would have had if the motion had been undisturbed are called perturbations.

The perturbative function is in this case, by equation (9.10),

$$(9.15) \quad P = Q \frac{R}{r} + G \frac{R \times (R \times S)}{\sqrt{[R \times (R \times S)]^2}} + H \frac{R \times S}{\sqrt{(R \times S)^2}},$$

and is here resolved into components along the three orthogonal directions R , $R \times (R \times S)$, and $R \times S$.

In this problem it is advantageous to resolve the disturbing acceleration into three rectangular components Q , F , and E , where Q is the component in the R direction, F is the component perpendicular to the plane of the orbit, and E is the component in the plane of the orbit, which acts at right angles to the radius vector with the positive direction making an angle less than 90° with the direction of motion. The resolution in this case would be

$$(9.16) \quad P = Q \frac{R}{r} + FN + E \frac{N \times R}{r}, \quad N^2 = 1,$$

where

¹⁵Moulton, Introduction to Celestial Mechanics, chap. v.

$$(9.17) \quad Q = (3/2r)K,$$

$$(9.18) \quad F = G \frac{N \cdot R \times (R \times S)}{\sqrt{[R \times (R \times S)]^2}} + H \frac{N \cdot R \times S}{\sqrt{(R \times S)^2}},$$

and

$$(9.19) \quad E = -G \frac{N \cdot R \times S}{\sqrt{(R \times S)^2}} + H \frac{N \cdot R \times (R \times S)}{\sqrt{[R \times (R \times S)]^2}}.$$

To express equation (9.16) in terms of the elements of the orbit we shall choose a set of rectangular axes x , y , and z with the origin at the center of mass O_1 of the spheroid. The directions of these axes are fixed so that the oz -axis has the same direction as the vector constant D of the moment of momentum integral, the unit vectors in these respective directions being i , j , and k . Let the plane of the orbit intersect the xy -plane in the line O_1L (Fig. 4), and let i represent the inclination between

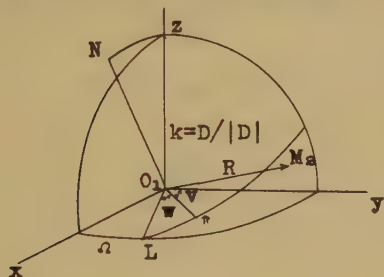


Fig. 4

the two planes. We shall designate the major semi-axis of the orbit by a , the eccentricity by e , the longitude of ascending node by Ω , the longitude of the perihelion point measured from the node by w , the time of perihelion passage by T , the angle between the perihelion point and R by v , and the angle $w + v$ by θ .

With these definitions we will proceed by expressing first the various terms needed for this transformation of the perturbative function in terms of the elements. Thus

$$R \cdot D = r|D| \sin \theta \sin i,$$

$$N \cdot D = |D| \cos i,$$

and

$$R \cdot (D \times N) = r|D| \cos \theta \sin i,$$

which can be seen directly from Fig. 4. From the case of the ordinary two-body problem referred to above we have the relations

$$r = \frac{a(1 - e^2)}{1 + e \cos v},$$

$$R \cdot R' = \frac{k e \sqrt{M_1 + M_2} \sqrt{a(1 - e^2)} \sin v}{1 + e \cos v},$$

$$\begin{aligned}
 R'^2 &= k^2(M_1 + M_2)\left(\frac{2}{r} - \frac{1}{a}\right), \\
 &= k^2(M_1 + M_2)\left(\frac{1 + 2e \cos v + e^2}{a(1 - e^2)}\right),
 \end{aligned}$$

and

$$R \times R' = N k \sqrt{M_1 + M_2} \sqrt{a(1 - e^2)}.$$

Then since, by (7.9),

$$S = MR \times R' - D$$

it follows that

$$S^2 = MM_1M_2k^2a(1 - e^2) - 2kM\sqrt{M_1 + M_2} \sqrt{a(1 - e^2)} |D| \cos i + D^2,$$

$$R \times S = \frac{Mke\sqrt{M_1 + M_2} \sqrt{a(1 - e^2)} \sin v}{1 + e \cos v} R - MR^2R' - R \times D,$$

$$(R \times S)^2 = R^2S^2 - R^2D^2 \sin^2 \theta \sin^2 i,$$

$$N \cdot R \times S = -N \cdot R \times D = -r |D| \cos \theta \sin i,$$

$$N \cdot R \times (R \times S) = -MR^2N \cdot (R \times R') + R^2N \cdot D$$

$$= MR^2k\sqrt{M_1 + M_2} \sqrt{a(1 - e^2)} + R^2 |D| \cos i,$$

$$K = k^2(M_1M_2)/a - S^2/A_1 + h.$$

If we let a_1 be the semi-major axis of the meridian section, and e_1 the eccentricity of the meridian section of the spheroid

$$C_1 - A_1 = M_1a_1^2e_1^2/5,$$

$$C_1 = 2M_1a_1^2/5,$$

and

$$A_1 = M_1a_1^2(2 - e_1^2)/5.$$

The components Q , F , and E of equation (9.16) can now be written in terms of the elements of the orbit as

$$(9.20) \quad Q = 3K/2r,$$

$$(9.21) \quad F = (f_1 \sin \theta - f_2) \frac{f_3}{f_4 + f_5 \sin^2 \theta}$$

$$\mp \sqrt{f_6 \sin^2 \theta + f_7 \sin \theta + f_8} \frac{r f_9 \cos \theta}{f_4 + f_5 \sin^2 \theta},$$

$$(9.22) \quad E = (f_1 \sin \theta - f_2) \frac{-f_9 \cos \theta}{f_4 + f_5 \sin^2 \theta} \\ + \sqrt{f_6 \sin^2 \theta + f_7 \sin \theta + f_8} \frac{r f_3}{f_4 + f_5 \sin^2 \theta},$$

where

$$(9.23) \quad f_1 = (K + \frac{k^2 M_1 M_2 a_1^2 e_1^2}{5r^3}) |D| r \sin i,$$

$$f_2 = \left[\frac{3k^2 M_1 M_2 a_1^2 e_1^2}{5r} (K + \frac{k^2 M_1 M_2 a_1^2 e_1^2}{5r^3}) \frac{2M_1 a_1^2}{5} b, \right.$$

$$f_3 = -kM \sqrt{M_1 + M_2} \sqrt{a(1 - e^2)} + |D| \cos i,$$

$$f_4 = R^2 S^2,$$

$$f_5 = -R^2 D^2 \sin^2 i,$$

$$f_6 = (K + \frac{k^2 M_1 M_2 e_1^2 a_1^2}{5r^3}) (\frac{-3D^2 \sin^2 i}{5r^3} \frac{k^2 M_1 M_2 e_1^2 a_1^2}{5r^3}),$$

$$f_7 = (K + \frac{k^2 M_1 M_2 e_1^2 a_1^2}{5r^3}) (\frac{4}{5} M_1 a_1^2 b |D| \sin i$$

$$\left[\frac{3k^2 M_1 M_2 a_1^2 e_1^2}{5r^3} \left(K + \frac{k^2 M_1 M_2 a_1^2 e_1^2}{5r^3} \right) \right],$$

$$f_8 = (K + \frac{k^2 M_1 M_2 e_1^2 a_1^2}{5r^3}) (\frac{k^2 M_1 M_2 a_1^2 e_1^2}{5r^3} 2S^2 - KS^2 - \frac{12k^2 M_1^3 M_2 a_1^6 e_1^2 b^2}{125r^3}),$$

$$f_9 = -|D| \sin i.$$

By substituting the above values of Q , F , and E in equation (9.16), we have the perturbative function expressed in terms of the elements of the orbit which was desired.

10. Variations of the Elements.—In section 9 we were able to resolve the disturbing force into three rectangular components all of which were expressed in terms of the elements of the instantaneous orbit. Now by making use of the equations for the variations of the elements in terms of the components Q , F , and E

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 worked out by Moulton¹⁶ we are able to write down at once these equations in terms of the elements of the problem considered here. Mean values of these rates will then be found for certain limiting cases.

The time rate of change of the ascending node

$$(10.1) \quad \frac{d\Omega}{dt} = \frac{r \sin \theta}{na^2 \sqrt{1-e^2} \sin i} F,$$

where

$$n = k \sqrt{M_1 + M_2} / a^2,$$

or, after substituting the value for F , equation (9.21),

$$\begin{aligned} \frac{d\Omega}{dt} = & (f_1 \sin \theta - f_2) \frac{f_3 r \sin \theta}{(f_4 - f_5 \sin^2 \theta) na^2 \sqrt{1-e^2} \sin i} \\ & + \sqrt{f_6 \sin^2 \theta + f_7 \sin \theta + f_8} \frac{r^2 f_9 \sin \theta \cos \theta}{(f_4 - f_5 \sin^2 \theta) na^2 \sqrt{1-e^2} \sin i}. \end{aligned}$$

If i and e are small quantities then f_1, f_5, f_6, f_7 , and f_9 are also small since they contain $\sin i$ as a factor as is seen by (9.23). Hence for i and e small

$$\begin{aligned} (10.2) \quad \frac{d\Omega}{dt} = & \frac{(K + k^2 M_1 M_2 a_1^2 e_1^2 / 5r^3) |D| f_3 r^2 \sin^2 \theta}{f_4 na^2 \sqrt{1-e^2}} \\ & - \frac{f_2 f_3 r \sin \theta}{f_4 na^2 \sqrt{1-e^2} \sin i} - \frac{\sqrt{f_8} f_9 r^2 \sin 2\theta}{2f_4 na^2 \sqrt{1-e^2} \sin i}. \end{aligned}$$

On taking the mean value over one revolution the second two terms in the right member of equation (10.2) are zero and the first term yields

$$(10.3) \quad \frac{d\Omega}{dt} = \frac{(K + k^2 M_1 M_2 a_1^2 e_1^2 / 5r^3) |D| f_3 r^2}{2f_4 na^2 \sqrt{1-e^2}},$$

where

$$\frac{d\Omega}{dt} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\Omega}{dt} d\theta.$$

Since, by (9.23) and (7.8),

¹⁶Moulton, An Introduction to Celestial Mechanics, p. 404.

$$f_3 = -kM\sqrt{M_1 + M_2} \sqrt{a(1 - e^2)} + |D| \cos i \\ = N \cdot [-M(R \times R') + D] = N \cdot [A_1 W_1 + (C_1 - A_1) b k_1]$$

and since, by (9.9),

$$(10.4) \quad (K + k^2 M_1 M_2 a_1^2 e_1^2 / 5r^3) = (3k^2 M_2 / r^5) (C_1 - A_1) (R \cdot k_1)^2 > 0,$$

and, by (9.23),

$$f_4 > 0,$$

then the value of $d\bar{\omega}/dt$ is greater or less than zero as $N \cdot [A_1 W_1 + (C_1 - A_1) b k_1]$ is greater or less than zero. Now if the projections $N \cdot W_1$, $N \cdot k_1$, and $b = W_1 \cdot k_1$ are positive then $N \cdot [A_1 W_1 + (C_1 - A_1) b k_1]$ is positive and the longitude of the ascending node will increase.

The time rate of change of the inclination

$$(10.5) \quad \frac{di}{dt} = \frac{r \cos \theta}{na^2 \sqrt{1 - e^2}} F,$$

or, by (9.21),

$$\frac{di}{dt} = (f_1 \sin \theta - f_2) \frac{rf_3 \cos \theta}{(f_4 + f_5 \sin^2 \theta) na^2 \sqrt{1 - e^2}} \\ - \sqrt{f_6 \sin^2 \theta + f_7 \sin \theta + f_8} \frac{r^2 f_9 \cos^2 \theta}{(f_4 + f_5 \sin^2 \theta) na^2 \sqrt{1 - e^2}}.$$

If i and e are small

$$(10.6) \quad \frac{di}{dt} = - \frac{f_2 f_3 r \cos \theta}{f_4 na^2 \sqrt{1 - e^2}},$$

and the mean value for one revolution is zero. The value of the inclination will then oscillate as the one body revolves about the other but up to the order of approximation considered here its mean change is zero.

The time rate of change of the perihelion point

$$(10.7) \quad \frac{dw}{dt} = - \frac{\sqrt{1 - e^2} \cos(\theta - w)}{nae} Q + \frac{\sqrt{1 - e^2}}{nae} [1 \\ + \frac{r}{a(1 - e^2)}] \sin(\theta - w) E - \frac{r \sin \theta \cot i}{na^2 \sqrt{1 - e^2}} F.$$

If i and e are small then, by (9.20), (9.21), (9.22), and (9.23),

$$\begin{aligned} \frac{dw}{dt} = & -\frac{3}{2} \frac{\sqrt{1-e^2} K \cos(\theta-w)}{naer} \\ & + \left(1 + \frac{r}{a(1-e^2)}\right) \left(-\frac{\sqrt{f_8} f_3 r}{f_4}\right) \frac{\sqrt{1-e^2} \sin(\theta-w)}{nae} \\ & - \left[\frac{(f_1 \sin \theta - f_2) f_3}{f_4} - \frac{\sqrt{f_8} f_3 r}{f_4}\right] \frac{r \sin \theta \cot i}{na^2 \sqrt{1-e^2}}, \end{aligned}$$

and the mean value for one revolution is

$$(10.8) \quad \frac{d\bar{w}}{dt} = - \frac{(K + k^2 M_1 M_2 a_1^2 e_1^2 / 5r^3) |D| f_3 r^2}{2f_4 na^2 \sqrt{1-e^2}}$$

which is the same as that for $d\bar{\pi}/dt$ except that it has the opposite sign. That is, as the node advances over a long period of time the perihelion point moves backward assuming that the initial conditions are such that the node is advancing.

The value of

$$(10.9) \quad \frac{de}{dt} = \frac{\sqrt{1-e^2} \sin(\theta-w)}{na} Q + \frac{\sqrt{1-e^2}}{na^2 e} \left[\frac{a^2(1-e^2)}{r} - r \right] E.$$

If i and e are small then, by (9.20), (9.22), and (9.23),

$$\frac{de}{dt} = \frac{3 \sin(\theta-w) K}{2nar} + \frac{2 \cos(\theta-w) \sqrt{f_8} f_3}{nf_4},$$

since when e is small

$$\frac{a^2(1-e^2) - r^2}{er} = \frac{2a^2 \cos v}{r}, \quad v = \theta - w,$$

or, within this degree of approximation, the mean value of de/dt is zero for one revolution. This means that the eccentricity, which defines the shape of the orbit, remains approximately the same over a long period of time.

The value of

$$\begin{aligned} (10.10) \quad \frac{d\sigma}{dt} = & -\frac{1}{na} \left[\frac{2r}{a} - \frac{1-e^2}{e} \cos(\theta-w) \right] Q \\ & - \frac{(1-e^2)}{nae} \left[1 + \frac{r}{a(1-e^2)} \right] \sin(\theta-w) E. \end{aligned}$$

On taking the mean value for one revolution if i and e are small, by (9.22),

$$\frac{d\sigma}{dt} = -\frac{2r}{na^2} Q$$

and, by (9.20),

$$(10.11) \quad \frac{d\sigma}{dt} = -\frac{3K}{na^2}.$$

Now K is negative, by (10.4), if the angle between N and k_1 is less than $\arccos \sqrt{3}/3$, therefore $d\sigma/dt$ is positive. Since $\sigma = -nT$, where n is the mean angular motion of the body and T the time of perihelion passage, the time of perihelion passage is decreasing.

The time rate of change of the major semi-axis

$$(10.12) \quad \frac{da}{dt} = \frac{2e \sin(\theta - w)}{n\sqrt{1-e^2}} Q + \frac{2a\sqrt{1-e^2}}{nr} E.$$

On taking the mean value for one revolution

$$\overline{\frac{da}{dt}} = \frac{2a\sqrt{1-e^2}}{nr} E,$$

and if i and e are small

$$(10.13) \quad \overline{\frac{da}{dt}} = \frac{-2a\sqrt{f_8} f_3}{nr_4}$$

after making use of (9.22) and (9.23).

Let q be the distance to the perihelion point. Then

$$q = a(1 - e),$$

and

$$(10.14) \quad \frac{dq}{dt} = -\frac{(1-e^2) \sin(\theta - w)}{n\sqrt{1-e^2}} Q + \frac{\sqrt{1-e^2}}{aenr} [r^2 - a^2(1-e^2)] E.$$

If i and e are small then, by (9.22) and (9.23),

$$\frac{dq}{dt} = -\frac{3K \sin(\theta - w)}{2nr} - \frac{2a\sqrt{f_8} f_3 [1 - \cos(\theta - w)]}{nr_4},$$

and the mean value for one revolution is

$$\overline{\frac{dq}{dt}} = -\frac{2a\sqrt{f_8} f_3}{nr_4}.$$

This is the same as the mean value for one revolution of da/dt

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which should be the case since $\overline{d\theta}/dt$ is zero within the degree of approximation considered here. The perihelion distance and the major semi-axis do not necessarily remain the same length over one revolution. For if f_8 is positive then these lengths become shorter which would happen in this case if

$$2s^2 - 3[C_1b]^2 > 0$$

thus making

$$f_8 = [k^2 M_2 (C_1 - A_1) / r^3 + K] [(k^2 M_2 [C_1 - A_1] / r^3) (2s^2 - 3[C_1b]^2) - s^2 K] > 0$$

since, by (10.4), the first factor is positive and, by (10.11), K is negative. For a particular problem these values might be determined and, therefore, the variations of a and q for their mean value of one revolution.

The results just obtained indicate, in the case of a spheroid and a sphere acting under their mutual attraction over a long period of time with the conditions assumed here, that the longitude of the ascending node increases; that the inclination of the orbit remains the same; that the longitude of the perihelion point measured from the node moves backward as the node advances; that the eccentricity remains approximately the same; that the time of perihelion passage decreases; and the major semi-axis does not necessarily remain the same length over a period of one revolution within the degree of approximation considered here.

The equations expressing the rates of variation of the elements of the elliptic orbit give a means of approximating the orbit of the body M_2 , which in this case is a sphere, about the body M_1 , which is a spheroid, when the spheroid differs only slightly from a sphere.

VITA

Samuel Schwartz Smith was born in Franklin, Idaho, October 26, 1892. He received his primary and secondary education in the public schools of Utah. He was granted a degree of Bachelor of Science in 1916, and the degree of Master of Science in 1917, both from the University of Utah. The following year he was made an instructor in mathematics at the University of Utah which position he resigned to attend the Military School of Aeronautics at Berkeley, California.

He graduated from this school in the summer of 1918. After attending the Mather Field at Sacramento, and March Field at Riverside, California, he was given a commission in the United States Army as second Lieutenant with flying status. He was again made an instructor in mathematics at the University of Utah in 1922, and continued as such until 1926. During the summer of 1924, he attended the University of California. From 1926 to 1928, he attended the University of Chicago.

Since 1928, he has been an instructor at the University of Utah, having attended the University of Chicago eight quarters during this time.

During his residence at the Universities of California and Chicago, he has had the privilege to study under Professors Moore, Slaughter, Dickson, Bliss, MacMillan, Lane, Graves, Lunn, C. V. L. Charlier, Birkhoff, Graustein, Hille, Bartky, and others. To all these he feels indebted, but especially to Professor Bartky, under whose direction this paper was prepared and who rendered valuable assistance in its preparation.

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